Subject Name: **Structural Analysis-I**

Subject Code: **CE-4005**

Semester: **4th**

Subject Name: Structure Analysis- I **Subject Code- CE4005**

UNIT-I

Virtual work and Energy Principles: Principles of Virtual work applied to deformable bodies, strain energy and complementary energy, Energy theorems, Maxwell's Reciprocal theorem, Analysis of Pin-Jointed frames for static loads.

Energy Methods in Structural Analysis

Virtual Work

Introduction

. From Castigliano's theorem it follows that for the statically determinate structure; the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load. In this lesson, the principle of virtual work is discussed. As compared to other methods, virtual work methods are the most direct methods for calculating deflections in statically determinate and indeterminate structures. This principle can be applied to both linear and nonlinear structures. The principle of virtual work as applied to deformable structure is an extension of the virtual work for rigid bodies. This may be stated as: if a rigid body is in equilibrium under the action of a system of forces and if it continues to remain in equilibrium if the body is given a small (virtual) displacement, then the virtual work done by the F–F–system of forces as 'it rides' along these virtual displacements is zero.

Complementary Strain Energy:

Consider the stress strain diagram as shown Fig 1.1. The area enclosed by the inclined line and the vertical axis is called the complementary strain energy. For a linearly elastic material the complementary strain energy and elastic strain energy are the same.

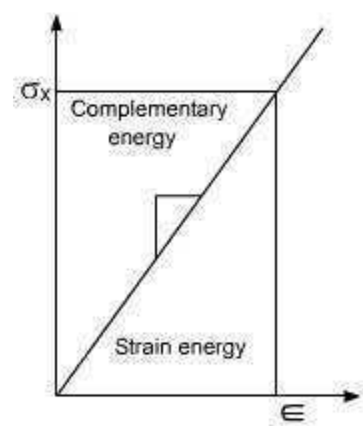


Figure 1.1: Stress strain diagram.

Let us consider elastic nonlinear prismatic bar subjected to an axial load. The resulting stress strain plot is as shown.

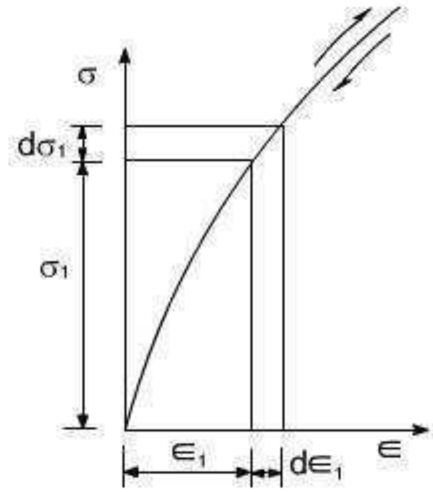


Figure 1.2: The resulting Stress strain diagram.

The new term complementary work is defined as follows

$$W^* = \int_0^P \delta_1 dP_1$$

we also know

$$\boxed{W^* + W = P\delta}$$

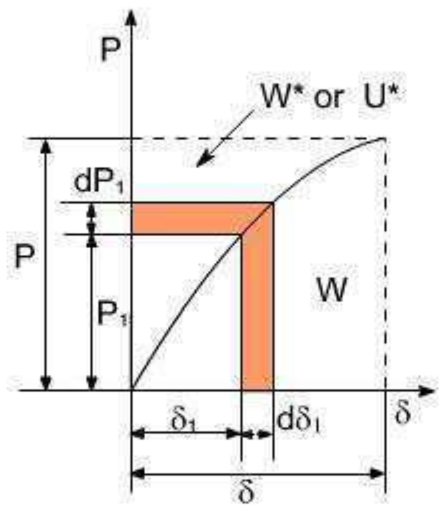


Figure 1.3: The resulting Stress strain diagram.

So in geometric sense the work W* is the complement of the work 'W' because it completes rectangle as shown in the above figure

Complementary Energy

$$U^* = W^* = \int_0^P \delta_1 dP$$

Likewise the complementary energy density u* is obtained by considering a volume element subjected to the stress s_1 and $\hat{I}_1$, in a manner analogous to that used in defining the strain energy density. Thus

$$U^* = \int_0^{\sigma} \epsilon_1 \, d\sigma 1$$

The complementary energy density is equal to the area between the stress strain curve and the stress axis. The total complementary energy of the bar may be obtained from u* by integration

$$U^* = \int dv$$

Sometimes the complementary energy is also called the stress energy. Complementary Energy is expressed in terms of the load and that the strain energy is expressed in terms of the displacement.

Castigliano's Theorem: Strain energy techniques are frequently used to analyze the deflection of beam and structures. Castigliano's theorem were developed by the Italian engineer Alberto Castigliano in the year 1873, these theorems are applicable to any structure for which the force deformation relations are linear

Castigliano's Theorem:

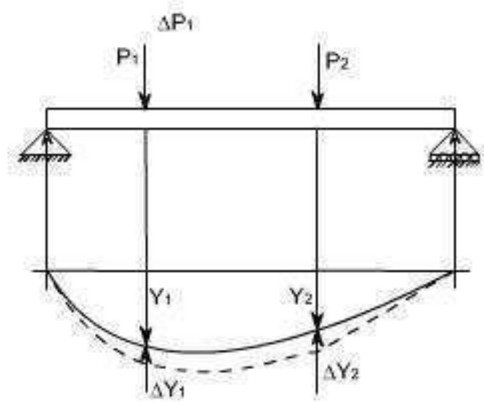


Figure 1.4: The loaded beam.

Consider a loaded beam as shown in figure 1.4

Let the two Loads P_1 and P_2 produce deflections Y_1 and Y_2 respectively strain energy in the beam is equal to the work done by the forces.

$$\boxed{U = \frac{1}{2}P_1Y_1 + \frac{1}{2}P_2Y_2} \qquad .....(1)$$

Let the Load P_1 be increased by an amount DP_1.

Let DP_1 and DP_2 be the corresponding changes in deflection due to change in load to DP_1.

Now the increase in strain energy $\Delta U = \frac{1}{2}\Delta P_1 \Delta Y_1 + P_1 \Delta Y_1 + P_2 \Delta Y_2 \qquad ....(2)$

Suppose the increment in load is applied first followed by P_1 and P_2 then the resulting strain energy is

$$\boxed{U+\Delta U = \frac{1}{2}\Delta P_1 \Delta Y_1 + \Delta P_1 Y_1 + P_2 \Delta Y_2 \frac{1}{2}P_1Y_1 + \frac{1}{2}P_2Y_2} \qquad ......(3)$$

Since the resultant strain energy is independent of order loading,

Combing equation 1, 2 and 3. One can obtain

$$\boxed{\Delta P_1 Y_1 = P_1 \Delta Y_1 + P_2 \Delta Y_2} \quad .....(4)$$

equations (2) and (4) can be combined to obtain

$$\boxed{\frac{\Delta U}{\Delta P_1} = y_1 + \frac{1}{2}\Delta Y_1} \quad .....(5)$$

or upon taking the limit as DP_1 approaches zero [Partial derivative are used because the strain energy is a function of both P_1 and P_2]

$$\boxed{\frac{\partial U}{\partial P} = Y_1} \quad ....(6)$$

For a general case there may be number of loads, therefore, the equation (6) can be written as

$$\boxed{\frac{\partial U}{\partial P_i} = Y_i} \quad ....(7)$$

The above equation is castigation's theorem:

The statement of this theorem can be put forth as follows; if the strain energy of a linearly elastic structure is expressed in terms of the system of external loads. The partial derivative of strain energy with respect to a concentrated external load is the deflection of the structure at the point of application and in the direction of that load.

In a similar fashion, Castigliano's theorem can also be valid for applied moments and resulting rotations of the structure

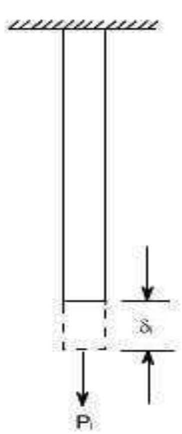
$$\boxed{\frac{\partial U}{\partial M_i} = \theta_i} \quad ....(8)$$

Where

M_i = applied moment

q_i = resulting rotation

Castigliano's First Theorem:



In similar fashion as discussed in previous section suppose the displacement of the structure are changed by a small amount dd$_i$. While all other displacements are held constant the increase in strain energy can be expressed as

$$\boxed{dU = \frac{\partial U}{\partial \delta_i} d\delta_i} \qquad ....(9)$$

Where

¶U / d$_i$ ® is the rate of change of the strain energy w.r.t d$_i$.

It may be seen that, when the displacement d$_i$ is increased by the small amount dd ; work done by the corresponding force only since other displacements are not changed.

The work which is equal to P$_i$dd$_i$ is equal to increase in strain energy stored in the structure

$$\boxed{du = P_i d\delta_i}$$

By rearranging the above expression, the Castigliano's first theorem becomes

$$\boxed{P_i = \frac{dU}{d\delta_i}}$$

The above relation states that the partial derivative of strain energy w.r.t. any displacement d$_i$ is equal to the corresponding force P$_i$ provided that the strain is expressed as a function of the displacements.

Maxwell-Betti Law of Reciprocal Deflections

Maxwell-Betti Law of real work is a basic theorem in the structural analysis. Using this theorem, it will be established that the flexibility coefficients in compatibility equations, formulated to solve indeterminate structures by the flexibility method, form a symmetric matrix and this will reduce the number of deflection computations. The Maxwell-Betti law also has applications in the construction of influence lines diagrams for statically indeterminate structures. The Maxwell-Betti law, which applies to any stable elastic structure (a beam, truss, or frame, for example) on unyielding supports and at constant temperature, states:

The deflection of point *A* in direction 1 due to unit load at point *B* in direction 2 is equal in the magnitude to the deflection of point *B* in direction 2 produced by a unit load applied at *A* in direction 1.

The Figure 4.31 explains the Maxwell-Betti Law of reciprocal displacements in which, the displacement $\Delta_{AB}\Delta$ is $_{BA}$ equal to the displacement.

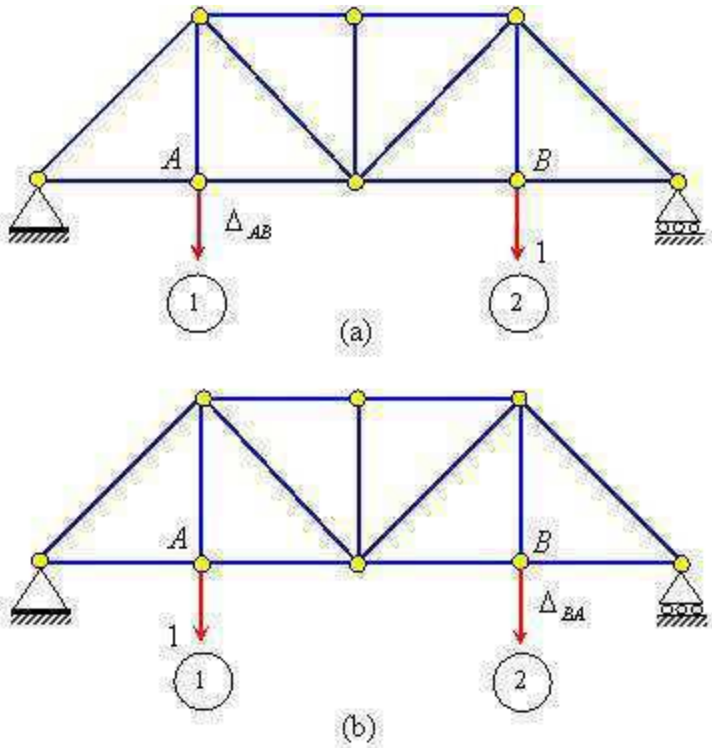


Figure 4.31 Illustration of Maxwell-Betti Law (directions 1 and 2 are shown by circle)

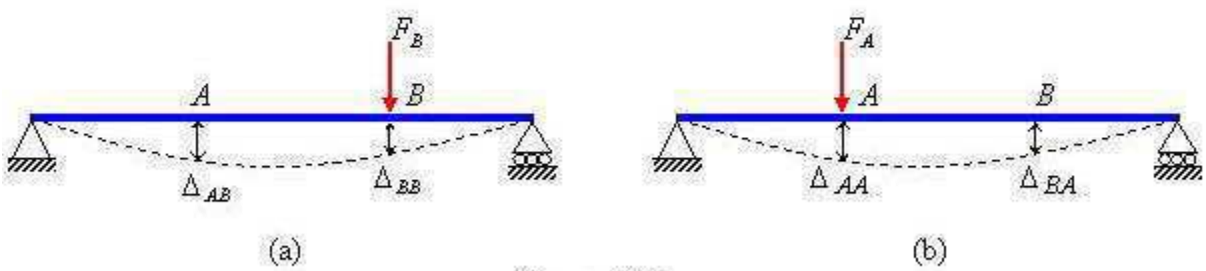


Figure 4.32

In order to prove the reciprocal theorem, consider the simple beam shown in Figure 4.32. Let a vertical force F_B at point *B* produces a vertical deflection Δ_{AB} at point *A* and Δ_{BB} at point *B* as shown in Figure 4.32(a). Similarly, in Figure 4.32(b) the application of a vertical force F_A at point *A* produces a vertical deflections Δ_{AA} and Δ_{BA} at points *A* and *B,* respectively. Let us evaluate the total work done by the two forces F_A and F_B when they are applied in different order to the zero to their final value.

Case 1: F_B applied and followed by F_A

(a) Work done when F_B is gradually applied

$$W_B = \frac{1}{2} F_B \Delta_{BB}$$ **s**

(b) **Work done when F_A is gradually applied with F_B in place**

$$W_A = \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA}$$

Total work done by the two forces for case 1 is

$$W_1 = W_B + W_A$$
$$= \frac{1}{2} F_B \Delta_{BB} + \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA}$$

Case2: F_A applied and followed by F_B

(c) Work done when F_A is gradually applied

$$W_A = \frac{1}{2} F_A \Delta_{AA}$$

(d) Work done when F_B is gradually applied with F_A in place

$$W_B = \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB}$$

Total work done by the two forces for case 2 is

$$W_2 = W_B + W_A$$
$$= \frac{1}{2} F_A \Delta_{AA} + \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB}$$

Since the final deflected position of the beam produced by the two cases of loads is the same regardless of the order in which the loads are applied. The total work done by the forces is also the same regardless of the order in which the loads are applied. Thus, equating the total work of Cases 1 and 2 give

$$W_1 = W_2$$

$$\frac{1}{2} F_B \Delta_{BB} + \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA} = \frac{1}{2} F_A \Delta_{AA} + \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB}$$

$$F_B \Delta_{BA} = F_A \Delta_{AB}$$

If $F_A = F_B = 1$, the equation (4.31) depicts the statement of the Maxwell-Betti law i.e.

$$\Delta_{BA} = \Delta_{AB}$$

The Maxwell-Betti theorem also holds for rotations as well as rotation and linear displacement in beams and frames.

Example 4.21 Verify Maxwell-Betti law of reciprocal displacement for the direction 1 and 2 of the pin-jointed structure shown in Figure (a).

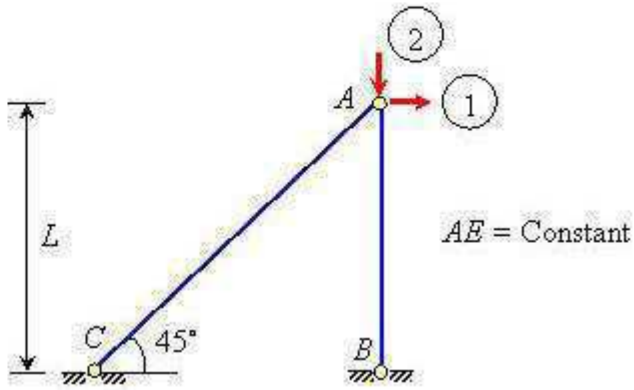


Figure 4.33(a)

Solution: Apply the forces P_1 and P_2 in the direction 1 and 2, respectively. The calculation of total strain energy in the system is given in Table 4.5.

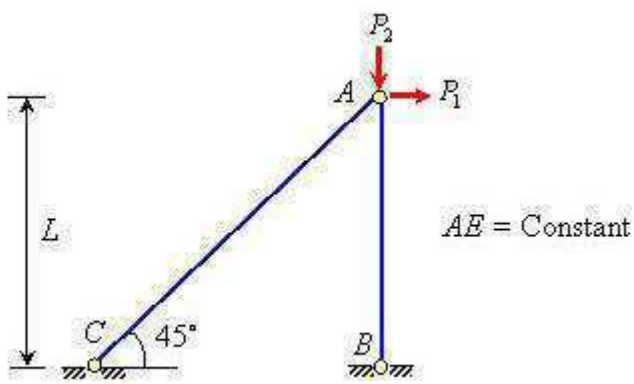


Figure 4.33(a)

Table 4.5

Membe	Length	$\sqrt{2}$ Force P	$U = P^2 L/2AE$
AB	L	$(P_1 + P_2)$	$(P_1+P_2)^2 L/2AE$
AC	$\sqrt{2}$ L	P 1	$\sqrt{2}P_1^2 L/AE$

$$\sum \ ((P_1+P_2)^2+2\sqrt{2}P_1^2)L/2AE$$

$$\Delta_{21}=\left.\frac{\partial U}{\partial P_2}\right|_{P_1=1,P_2=0}$$

$$=\left.(2(P_1+P_2)1+0)\frac{L}{2AE}\right|_{P_1=1,P_2=0}$$

$$=\frac{L}{AE}$$

$$\Delta_{12}=\left.\frac{\partial U}{\partial P_1}\right|_{P_1=0,P_2=1}$$

$$=\left.(2(P_1+P_2)1+4\sqrt{2}P_1)\frac{L}{2AE}\right|_{P_1=0,P_2=1}$$

$$=\frac{L}{AE}$$

Since $\Delta_{12}=\Delta_{21}$, hence the Maxwell-Betti law of reciprocal displacement is proved.

Example: Verify Maxwell-Betti law of reciprocal displacement for the cantilever beam shown in Figure 4.34(a).

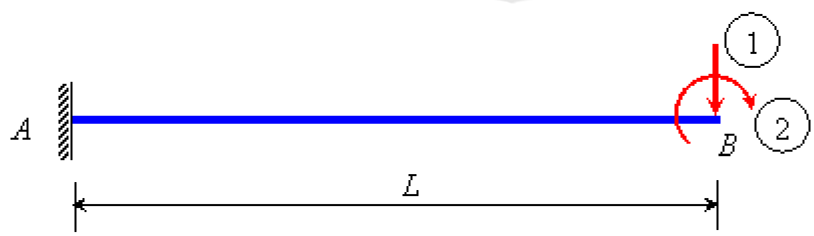


Figure 4.34(a)

Solution: Apply the forces P_1 and P_2 in the directions 1 and 2, respectively. The total strain energy stored is calculated below.

Consider any point *X* at a distance *x* from *B.*

$$M_x=-(P_1x+P_2)$$

$$U=\int_0^L\frac{M_x^2dx}{2EI}$$

$$=\frac{1}{2EI}\int_0^L(P_1x+P_2)^2dx$$

$$=\frac{1}{2EI}\left(\frac{P_1^2L^3}{3}+P_1P_2L^2+P_2^2L\right)$$

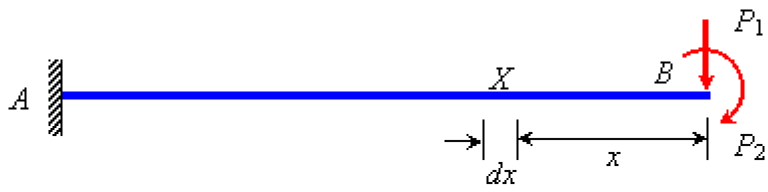


Figure 4.34(b)

$$\Delta_{12} = \left.\frac{\partial U}{\partial P_1}\right|_{P_1=0, P_2=1}$$

$$= \frac{1}{2EI}\left.\left(\frac{2P_1L^3}{3} + P_2L^2 + 0\right)\right|_{P_1=0, P_2=1}$$

$$= \frac{L^2}{2EI}$$

$$\Delta_{21} = \left.\frac{\partial U}{\partial P_2}\right|_{P_1=1, P_2=0}$$

$$= \left.\frac{1}{2EI}(0 + P_1L^2 + 2P_2L)\right|_{P_1=1, P_2=0}$$

$$= \frac{L^2}{2EI}$$

Since $\Delta_{12} = \Delta_{21}$**, the Maxwell-Betti law of reciprocal displacement is proved.**

Example: Verify Maxwell-Betti law of reciprocal displacement for the rigid-jointed plane frame with reference to marked direction as shown in Figure 4.35(a). ***EI*** **is same for both members.**

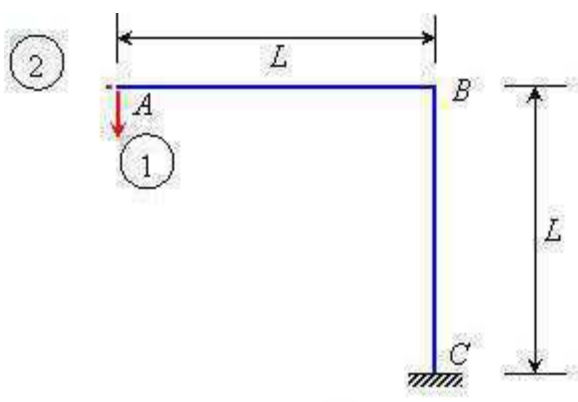


Figure 4.35(a)

Solution: Apply the forces P_1 **and** P_2 **in the directions 1 and 2, respectively as shown in Figure (b).**

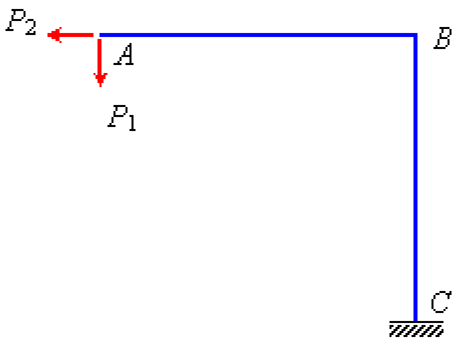


Figure 4.35(b)

Consider *AB:* (x measured from *A)*

$$M_x = -P_1 x$$

$$U_{AB} = \int_0^L \frac{M_x^2 dx}{2EI}$$

$$= \frac{1}{2EI}\int_0^L (-P_1 x)^2 dx$$

$$= \frac{P_1^2 L^3}{6EI}$$

Consider *BC:* (x measured from *B)*

$$M_x = -P_1 L - P_2 x$$

$$U_{BC} = \int_0^L \frac{M_x^2 dx}{2EI}$$

$$= \frac{1}{2EI}\int_0^L (-P_1 L - P_2 x)^2 dx$$

$$= \frac{P_1^2 L^3}{2EI} + \frac{P_1 P_2 L^3}{2EI} + \frac{P_2^2 L^3}{6EI}$$

Thus

$$U = U_{AB} + U_{BC}$$

$$= \frac{P_1^2 L^3}{6EI} + \frac{P_1^2 L^3}{2EI} + \frac{P_1 P_2 L^3}{2EI} + \frac{P_2^2 L^3}{6EI}$$

$$= \frac{L^3}{6EI}(4P_1^2 + 3P_1 P_2 + P_2^2)$$

The displacement in the direction 1 due to unit load applied in 2 is

$$\Delta_{12} = \left.\frac{\partial U}{\partial P_1}\right|_{P_1=0,P_2=1}$$

$$= \frac{L^3}{6EI}(8P_1 + 3P_2 + 0)\Big|_{P_1=0,P_2=1}$$

$$= \frac{L^3}{2EI}$$

The displacement in the direction 2 due to unit load applied in 1 is

$$\Delta_{21} = \left.\frac{\partial U}{\partial P_2}\right|_{P_1=1,P_2=0}$$

$$= \frac{L^3}{6EI}(0 + 3P_1 + 2P_2)\Big|_{P_1=1,P_2=0}$$

$$= \frac{L^3}{2EI}$$

Since $\Delta_{12} = \Delta_{21}$, proves the Maxwell-Betti law of reciprocal displacements.

